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ANALYTICAL SOLUTION OF A REFINED SHEAR DEFORMATION THEORY FOR RECTANGULAR COMPOSITE PLATES

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Abstract—The Lévy type solution procedure in conjunction with the state-space concept is used to determine the deflections and stresses for symmetric laminated composite plates with rectangular geometries by using a refined shear deformation theory. Combinations of simply supported, clamped and free boundary conditions are considered. Numerical results are presented for rectangular plates with different edge conditions, aspect ratios, lamination schemes and loadings. The solution should serve as a reference for designers and practitioners of numerical/computational methods.

INTRODUCTION

Three-dimensional elasticity solutions for the bending of simply supported thick orthotropic rectangular plates and laminates were obtained by Srinivas and his coworkers[1–3], and Pagano[4]. The Navier solution of simply supported rectangular plates was developed by Whitney and Leissa[5] for classical laminate theory, Pagano and Whitney[6–8], Bert and Chen[9] and Reddy and Chao[10] for the first-order shear deformation (i.e. the Reissner–Mindlin plate) theory[11], and by Reddy and his colleagues[12–14] for refined shear deformation theories. Recently, the Lévy type solutions were developed by Reddy *et al.*[15] for symmetric laminates with different combinations of free, clamped and simply supported boundary conditions by using the first-order shear deformation theory.

The present study deals with the development of the Lévy type solution of the refined shear deformation theory of Reddy[12, 13] for symmetric rectangular laminates with two opposite edges simply supported and the remaining edges subjected to a combination of free, simply supported and clamped boundary conditions. The state-space concept is used to solve the ordinary differential equations obtained after the application of the Lévy solution procedure.

GOVERNING EQUATIONS

Consider a laminated plate composed of N orthotropic layers, symmetrically located with respect to the midplane of the laminate. The governing equations of the refined theory are based on the following displacement field[11–13]:

$$u_1 = u+z\left[\psi_x-\frac{4}{3}\left(\frac{z}{h}\right)^2\left(\psi_x+\frac{\partial w}{\partial x}\right)\right]$$

$$u_2 = v+z\left[\psi_y-\frac{4}{3}\left(\frac{z}{h}\right)^2\left(\psi_y+\frac{\partial w}{\partial y}\right)\right]$$

$$u_3 = w \tag{1}$$

where (u_1, u_2, u_3) are the displacements along the x-, y- and z-coordinates respectively (u, v, w) are the corresponding displacements of a point on the midplane of the laminate, and ψ_x and ψ_y are the rotations of a transverse normal about the y- and x-axes, respectively.

The cubic variation of u_1 and u_2 through laminate thickness introduces higher-order resultants

$$P_i = \int_{-h/2}^{h/2} \sigma_i z^3 \, dz \qquad (i = 1, 2, 6)$$

$$(R_1, R_2) = \int_{-h/2}^{h/2} z^2(\sigma_5, \sigma_4) \, dz$$

and laminate stiffnesses

$$(F_{ij}, H_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(z^4, z^6) \, dz \qquad (i, j = 1, 2, 6)$$

$$(D_{ij}, F_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(z^2, z^4) \, dz \qquad (i, j = 4, 5).$$

For symmetrical cross-ply laminated plates, the following stiffness coefficients vanish[11]:

$$B_{ij} = E_{ij} = 0 \qquad \text{for} \quad i, j = 1, 2, 4, 5, 6$$
$$A_{16} = A_{26} = D_{16} = D_{26} = F_{16} = F_{26} = H_{16} = H_{26} = 0$$
$$A_{45} = D_{45} = F_{45} = 0.$$

This implies that the effect of coupling between stretching and bending vanishes. For such laminates the governing equations are given by (see Refs [11, 12])

$$\begin{aligned}
&\frac{4}{3h^2}\left[F_{11}\frac{\partial^3\psi_x}{\partial x^3} + H_{11}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^3\psi_x}{\partial x^3} + \frac{\partial^4 w}{\partial x^4}\right) + F_{12}\frac{\partial^3\psi_y}{\partial x^2\,\partial y} + H_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^3\psi_y}{\partial x^2\,\partial y}\right.\right.\\
&\left. + \frac{\partial^4 w}{\partial x^2\,\partial y^2}\right) + F_{12}\frac{\partial^3\psi_x}{\partial y^2\,\partial x} + H_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^3\psi_x}{\partial y^2\,\partial x} + \frac{\partial^4 w}{\partial x^2\,\partial y^2}\right) + F_{22}\frac{\partial^3\psi_y}{\partial y^3}\\
&+ H_{22}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^3\psi_y}{\partial y^3} + \frac{\partial^4 w}{\partial y^4}\right) + 2F_{66}\left(\frac{\partial^3\psi_y}{\partial x^2\,\partial y} + \frac{\partial^3\psi_x}{\partial y^2\,\partial x}\right) + 2H_{66}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^3\psi_x}{\partial y^2\,\partial x}\right.\\
&\left.\left. + \frac{\partial^3\psi_y}{\partial x^2\,\partial y} + \frac{2\partial^4 w}{\partial x^2\,\partial y^2}\right)\right] - \frac{4}{h^2}\left[D_{55}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial\psi_x}{\partial x}\right) + F_{55}\left(-\frac{4}{h^2}\right)\left(\frac{\partial\psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right)\right.\\
&\left. + D_{44}\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial\psi_y}{\partial y}\right) + F_{44}\left(-\frac{4}{h^2}\right)\left(\frac{\partial\psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right)\right] + \left[A_{55}\left(\frac{\partial\psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right)\right.\\
&\left. + D_{55}\left(-\frac{4}{h^2}\right)\left(\frac{\partial\psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + A_{44}\left(\frac{\partial\psi_y}{\partial y}\frac{\partial^2 w}{\partial y^2}\right) + D_{44}\left(-\frac{4}{h^2}\right)\left(\frac{\partial\psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right)\right] + q = 0 \qquad \text{(2a)}
\end{aligned}$$

$$\begin{aligned}
&D_{11}\frac{\partial^2\psi_x}{\partial x^2} + D_{12}\frac{\partial^2\psi_y}{\partial x\,\partial y} + F_{11}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_x}{\partial x^2} + \frac{\partial^3 w}{\partial x^3}\right) + F_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_y}{\partial x\,\partial y}\right.\\
&\left. + \frac{\partial^3 w}{\partial x\,\partial y^2}\right) + D_{66}\left(\frac{\partial^2\psi_x}{\partial y^2} + \frac{\partial^2\psi_y}{\partial x\,\partial y}\right) + F_{66}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_x}{\partial y^2} + \frac{\partial^2\psi_y}{\partial x\,\partial y} + \frac{2\partial^3 w}{\partial x\,\partial y^2}\right)\\
&- \left[A_{55}\left(\psi_x + \frac{\partial w}{\partial x}\right) + D_{55}\left(-\frac{4}{h^2}\right)\left(\psi_x + \frac{\partial w}{\partial x}\right)\right] - \frac{4}{3h^2}\left[F_{11}\frac{\partial^2\psi_x}{\partial x^2}\right.\\
&+ H_{11}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_x}{\partial x^2} + \frac{\partial^3 w}{\partial x^3}\right) + F_{12}\frac{\partial^2\psi_y}{\partial x\,\partial y} + H_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_y}{\partial x\,\partial y} + \frac{\partial^3 w}{\partial x\,\partial y^2}\right)
\end{aligned}$$

$$+F_{66}\left(\frac{\partial^2\psi_y}{\partial x\,\partial y}+\frac{\partial^2\psi_x}{\partial y^2}\right)+H_{66}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_x}{\partial y^2}+\frac{\partial^2\psi_y}{\partial x\,\partial y}+\frac{2\partial^3 w}{\partial x\,\partial y^2}\right)\Bigg]+\frac{4}{h^2}\left[D_{55}\left(\frac{\partial w}{\partial x}+\psi_x\right)\right.$$

$$\left.+F_{55}\left(-\frac{4}{h^2}\right)\left(\psi_x+\frac{\partial w}{\partial x}\right)\right]=0 \tag{2b}$$

$$D_{66}\left(\frac{\partial^2\psi_x}{\partial x\,\partial y}+\frac{\partial^2\psi_y}{\partial x^2}\right)+F_{66}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_x}{\partial x\,\partial y}+\frac{\partial^2\psi_y}{\partial x^2}+2\frac{\partial^3 w}{\partial x^2\,\partial y}\right)+D_{12}\frac{\partial^2\psi_x}{\partial x\,\partial y}$$

$$+D_{22}\frac{\partial^2\psi_y}{\partial y^2}+F_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_x}{\partial x\,\partial y}+\frac{\partial^3 w}{\partial x^2\,\partial y}\right)+F_{22}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_y}{\partial y^2}+\frac{\partial^3 w}{\partial y^3}\right)$$

$$-\left[A_{44}\left(\psi_y+\frac{\partial w}{\partial y}\right)+D_{44}\left(-\frac{4}{h^2}\right)\left(\psi_y+\frac{\partial w}{\partial y}\right)\right]-\frac{4}{3h^2}\left[F_{66}\left(\frac{\partial^2\psi_y}{\partial x^2}+\frac{\partial^2\psi_x}{\partial y\,\partial x}\right)\right.$$

$$+H_{66}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_x}{\partial x\,\partial y}+\frac{\partial^2\psi_y}{\partial x^2}+\frac{2\partial^3 w}{\partial x^2\,\partial y}\right)+F_{12}\frac{\partial^2\psi_x}{\partial x\,\partial y}+H_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_x}{\partial x\,\partial y}\right.$$

$$\left.\left.+\frac{\partial^3 w}{\partial x^2\,\partial y}\right)+F_{22}\frac{\partial^2\psi_y}{\partial y^2}+H_{22}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial^2\psi_y}{\partial y^2}+\frac{\partial^3 w}{\partial y^3}\right)\right]+\frac{4}{h^2}\left[D_{44}\left(\frac{\partial w}{\partial y}+\psi_y\right)\right.$$

$$\left.+F_{44}\left(-\frac{4}{h^2}\right)\left(\frac{\partial w}{\partial y}+\psi_y\right)\right]=0. \tag{2c}$$

Here w denotes the transverse displacement, ψ_x and ψ_y are the rotations of the normal to midplane about the y- and x-axes, respectively, q is the distributed transverse load, and A_{ij}, D_{ij}, F_{ij}, H_{ij} are the plate stiffnesses, defined by

$$(D_{ij}, F_{ij}, H_{ij}) = \sum_{k=1}^{N}\int_{z_k}^{z_{k+1}} Q_{ij}^{(k)}(z^2, z^4, z^6)\,\mathrm{d}z \qquad (i, j = 1, 2, 6)$$

$$(A_{ij}, D_{ij}, F_{ij}) = \sum_{k=1}^{N}\int_{z_k}^{z_{k+1}} Q_{ij}^{(k)}(1, z^2, z^4)\,\mathrm{d}z \qquad (i, j = 4, 5). \tag{3}$$

Here $Q_{ij}^{(k)}$ denote the reduced orthotropic moduli of the kth lamina. The boundary conditions of the refined theory are of the form: specify

$$\left.\begin{array}{lll} w & \text{or} & Q_n \\ \dfrac{\partial w}{\partial n} & \text{or} & P_n \\ \psi_n & \text{or} & M_n \\ \psi_{ns} & \text{or} & M_{ns} \end{array}\right\} \quad \text{on} \quad \Gamma \tag{4}$$

where Γ is the boundary of the midplane Ω of the plate, and

$$\begin{aligned} M_n &= \hat{M}_1 n_x^2+\hat{M}_2 n_y^2+2\hat{M}_6 n_x n_y \\ M_{ns} &= (\hat{M}_2-\hat{M}_1)n_x n_y+\hat{M}_6(n_x^2-n_y^2) \\ P_n &= P_1 n_x^2+P_2 n_y^2+2P_6 n_x n_y \\ P_{ns} &= (P_2-P_1)n_x n_y+P_6(n_x^2-n_y^2) \\ Q_n &= \hat{Q}_1 n_x+\hat{Q}_2 n_y+\frac{4}{3h^2}\left(\frac{\partial P_{ns}}{\partial s}+\frac{\partial P_n}{\partial n}\right) \end{aligned}$$

$$\hat{M}_i = M_i - \frac{4}{3h^2}P_i \qquad (i = 1, 2, 6)$$

$$\hat{Q}_i = Q_i - \frac{4}{h^2}R_i \qquad (i = 1, 2)$$

$$\frac{\partial}{\partial n} = n_x\frac{\partial}{\partial x} + n_y\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial s} = n_x\frac{\partial}{\partial y} - n_y\frac{\partial}{\partial x}. \tag{5}$$

The stress resultants appearing in eqns (5) can be expressed in terms of the generalized displacements (w, ψ_x, ψ_y) as

$$M_1 = D_{11}\frac{\partial\psi_x}{\partial x} + D_{12}\frac{\partial\psi_y}{\partial y} + F_{11}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + F_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_2 = D_{12}\frac{\partial\psi_x}{\partial x} + D_{22}\frac{\partial\psi_y}{\partial y} + F_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + F_{22}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_6 = D_{66}\left(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x}\right) + F_{66}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x\,\partial y}\right)$$

$$Q_2 = A_{44}\left(\psi_y + \frac{\partial w}{\partial y}\right) + D_{44}\left(-\frac{4}{h^2}\right)\left(\psi_y + \frac{\partial w}{\partial y}\right)$$

$$Q_1 = A_{55}\left(\psi_x + \frac{\partial w}{\partial x}\right) + D_{55}\left(-\frac{4}{h^2}\right)\left(\psi_x + \frac{\partial w}{\partial x}\right)$$

$$P_1 = F_{11}\frac{\partial\psi_x}{\partial x} + F_{12}\frac{\partial\psi_y}{\partial y} + H_{11}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + H_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$P_2 = F_{12}\frac{\partial\psi_x}{\partial x} + F_{22}\frac{\partial\psi_y}{\partial y} + H_{12}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + H_{22}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$P_6 = F_{66}\left(\frac{\partial\psi_y}{\partial x} + \frac{\partial\psi_x}{\partial y}\right) + H_{66}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x\,\partial y}\right)$$

$$R_2 = D_{44}\left(\frac{\partial w}{\partial y} + \psi_y\right) + F_{44}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial w}{\partial y} + \psi_y\right)$$

$$R_1 = D_{55}\left(\frac{\partial w}{\partial x} + \psi_x\right) + F_{55}\left(-\frac{4}{3h^2}\right)\left(\frac{\partial w}{\partial x} + \psi_x\right). \tag{6}$$

THE SOLUTION PROCEDURE

The Lévy method can be used to solve eqns (2) for rectangular plates for which two opposite edges are simply supported. The other two edges can each have arbitrary boundary conditions. Here we assume that the edges parallel to the y-axis are simply supported, and the origin of the coordinate system is taken as shown in Fig. 1. The simply supported boundary conditions can be satisfied by trigonometric functions in x. The resulting ordinary differential equations in y can be solved using the state-space concept.

Following the Lévy type procedure, we assume the following representation of the

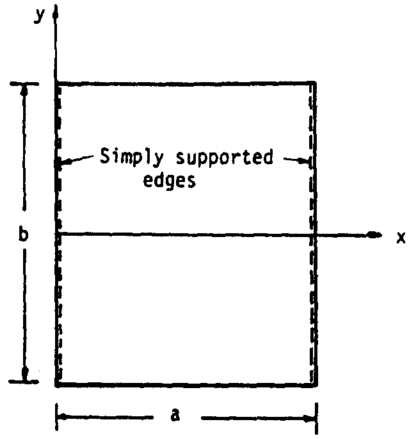

Fig. 1. Geometry and coordinate system of rectangular plate.

displacements and loading:

$$w(x, y) = \sum_{m=1}^{\infty} W_m(y) \sin \alpha x$$

$$\psi_x(x, y) = \sum_{m=1}^{\infty} X_m(y) \cos \alpha x$$

$$\psi_y(x, y) = \sum_{m=1}^{\infty} Y_m(y) \sin \alpha x$$

$$q(x, y) = \sum_{m=1}^{\infty} Q_m(y) \sin \alpha x \tag{7}$$

where $\alpha = m\pi/a$ and W_m, X_m, Y_m and Q_m denote amplitudes of w, ψ_x, ψ_y and q, respectively. Substituting eqns (7) into eqns (2), we obtain

$$e_1 W_m'''' + e_2 W_m'' + e_3 W_m + e_4 X_m'' + e_5 X_m + e_6 Y_m''' + e_7 Y_m' + Q_m = 0$$

$$e_8 W_m'' + e_9 W_m + e_{10} X_m'' + e_{11} X_m + e_{12} Y_m' = 0$$

$$e_{13} W_m''' + e_{14} W_m' + e_{15} X_m' + e_{16} Y_m'' + e_{17} Y_m = 0 \tag{8}$$

where primes on the variables indicate differentiation with respect to y, and

$$e_1 = -\left(\frac{4}{3h^2}\right)^2 H_{22}, \qquad e_2 = 2\left(\frac{4}{3h^2}\right)^2 \alpha^2 (H_{12} + 2H_{66}) + A_{44} - \frac{8}{h^2} D_{44} + \left(\frac{4}{h^2}\right)^2 F_{44}$$

$$e_3 = -\alpha^2 \left[\left(\frac{4}{3h^2}\right)^2 \alpha^2 H_{11} + \left(\frac{4}{h^2}\right)^2 F_{55} - \frac{8}{h^2} D_{55} + A_{55}\right]$$

$$e_4 = \alpha \frac{4}{3h^2}\left[-F_{12} + \frac{4}{3h^2} H_{12} - 2F_{66} + \frac{8}{3h^2} H_{66}\right]$$

$$e_5 = \alpha^3 \frac{4}{3h^2}\left(F_{11} - \frac{4}{3h^2} H_{11}\right) + \alpha\left[\frac{8}{h^2} D_{55} - \left(\frac{4}{h^2}\right)^2 F_{55} - A_{55}\right]$$

$$e_6 = \frac{4}{3h^2}\left(F_{22} - \frac{4}{3h^2} H_{22}\right)$$

$$e_7 = \alpha^2 \frac{4}{3h^2}\left[-F_{12} - 2F_{66} + \frac{4}{3h^2}(H_{12} + 2H_{66})\right] - \frac{8}{h^2} D_{44} + \left(\frac{4}{h^2}\right)^2 F_{44} + A_{44}$$

$$e_8 = e_4, \quad e_9 = e_5, \quad e_{10} = D_{66} - \frac{8}{3h^2}F_{66} + \left(\frac{4}{3h^2}\right)^2 H_{66}$$

$$e_{11} = \alpha^2\left[-D_{11} + \frac{8}{3h^2}F_{11} - \left(\frac{4}{3h^2}\right)^2 H_{11}\right] + \frac{8}{h^2}D_{55} - \left(\frac{4}{h^2}\right)^2 F_{55} - A_{55}$$

$$e_{12} = \alpha\left[D_{12} + D_{66} - \frac{8}{3h^2}(F_{12}+F_{66}) + \left(\frac{4}{3h^2}\right)^2 (H_{12}+H_{66})\right]$$

$$e_{13} = -e_6, \quad e_{14} = -e_7, \quad e_{15} = -e_{12}, \quad e_{16} = D_{22} - \frac{8}{3h^2}F_{22} + \left(\frac{4}{3h^2}\right)^2 H_{22}$$

$$e_{17} = \alpha^2\left[-D_{66} + \frac{8}{3h^2}F_{66} - \left(\frac{4}{3h^2}\right)^2 H_{66}\right] + \frac{8}{h^2}D_{44} - \left(\frac{4}{h^2}\right)^2 F_{44} - A_{44}. \tag{9}$$

Equations (8) can be written as

$$\begin{aligned} W_m'''' &= c_1 W_m'' + c_2 W_m + c_3 X_m + c_4 Y_m' + c_0 Q_m \\ X_m'' &= c_5 W_m'' + c_6 W_m + c_7 X_m + c_8 Y_m' \\ Y_m'' &= c_9 W_m''' + c_{10} W_m' + c_{11} X_m' + C_{12} Y_m \end{aligned} \tag{10}$$

where

$$\begin{aligned} c_1 &= \left(\frac{e_4^2}{e_{10}} + \frac{e_4 e_6 e_{12}}{e_{10} e_{16}} - \frac{e_6 e_7}{e_{16}} - e_2\right) \Big/ \left(e_1 + \frac{e_6^2}{e_{16}}\right) \\ c_2 &= \left(\frac{e_4 e_5}{e_{10}} + \frac{e_5 e_6 e_{12}}{e_{10} e_{16}} - e_3\right) \Big/ \left(e_1 + \frac{e_6^2}{e_{16}}\right) \\ c_3 &= \left(\frac{e_{11} e_4}{e_{10}} + \frac{e_{11} e_6 e_{12}}{e_{10} e_{16}} - e_5\right) \Big/ \left(e_1 + \frac{e_6^2}{e_{16}}\right) \\ c_4 &= \left(\frac{e_6 e_{17}}{e_{16}} + \frac{e_4 e_{12}}{e_{10}} + \frac{e_6 e_{12}^2}{e_{10} e_{16}} - e_7\right) \Big/ \left(e_1 + \frac{e_6^2}{e_{16}}\right) \\ c_0 &= -\frac{e_{16}}{e_1 e_{16} + e_6^2} \\ c_5 &= -e_4/e_{10}, \quad c_6 = -e_5/e_{10}, \quad c_7 = -e_{11}/e_{10}, \quad c_8 = -e_{12}/e_{10} \\ c_9 &= e_6/e_{16}, \quad c_{10} = e_7/e_{16}, \quad c_{11} = e_{12}/e_{16}, \quad c_{12} = -e_{17}/e_{16}. \end{aligned} \tag{11}$$

The linear system of ordinary differential equations, eqns (10), with constant coefficients can be reduced to a single matrix differential equation using the state-space concept (see Ref. [16])

$$\mathbf{x}' = A\mathbf{x} + \mathbf{b}. \tag{12}$$

This can be done by introducing the variables

$$\begin{aligned} &x_1 = W_m, \quad x_2 = W_m', \quad x_3 = W_m'', \quad x_4 = W_m''' \\ &x_5 = X_m, \quad x_6 = X_m', \quad x_7 = Y_m, \quad x_8 = Y_m' \end{aligned} \tag{13}$$

where

$$\mathbf{x}' = \begin{Bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \\ x_6' \\ x_7' \\ x_8' \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ C_0 Q_m \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ c_2 & 0 & c_1 & 0 & c_3 & 0 & 0 & c_4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ c_6 & 0 & c_5 & 0 & c_7 & 0 & 0 & c_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & c_{10} & 0 & c_9 & 0 & c_{11} & c_{12} & 0 \end{bmatrix}. \tag{14}$$

The solution of eqn (12) is given by

$$\mathbf{x} = \mathrm{e}^{Ay}\,\mathbf{K} + \mathrm{e}^{Ay}\int \mathrm{e}^{-A\xi}\,\mathbf{b}\,\mathrm{d}\xi \tag{15}$$

where $\mathbf{K}$ is a constant vector to be determined from the boundary conditions, e^{Ay} denotes the product

$$\mathrm{e}^{Ay} = [c] \begin{bmatrix} \mathrm{e}^{\lambda_1 y} & & & \\ & \mathrm{e}^{\lambda_2 y} & & \mathbf{0} \\ \mathbf{0} & & \ddots & \\ & & & \mathrm{e}^{\lambda_8 y} \end{bmatrix} [c]^{-1} \tag{16}$$

$[c]$ is the matrix of eigenvectors, $\lambda_i (i = 1, 2, 3, \ldots, 8)$ are the distinct eigenvalues associated with matrix A, and $[c]^{-1}$ is the inverse of the eigenvectors matrix $[c]$.

The following boundary conditions are used on the remaining two edges (i.e. the edges parallel to the x-axis) at $y = \mp b/2$:

simply supported $\quad w = \psi_x = P_2 = M_2 = 0$

clamped $\quad w = \dfrac{\partial w}{\partial y} = \psi_x = \psi_y = 0$

free $\quad P_2 = M_2 = 0$

$$M_6 - \frac{4}{3h^2} P_6 = 0$$

$$Q_2 - \frac{4}{h^2} R_2 + \frac{4}{3h^2}\left(\frac{\partial P_6}{\partial x} + \frac{\partial P_2}{\partial y}\right) = 0. \tag{17}$$

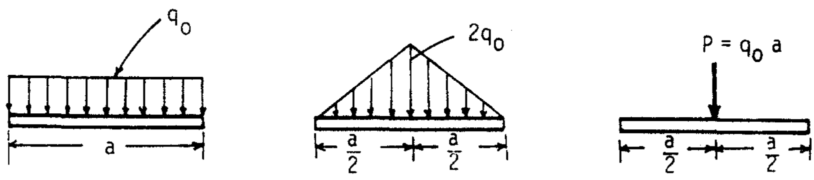

Fig. 2. Various types of transverse loads.

NUMERICAL RESULTS

Numerical results are presented for orthotropic and symmetric cross-ply (0°/90°/0°) plates subjected to three types of loads: uniformly distributed load (q_0), triangular distributed load ($2q_0$) and concentrated load P as shown in Fig. 2. The following sets of material properties are used in the calculations:

Material I

$$E_1 = 20.83 \times 10^6 \text{ psi}, \quad E_2 = 10.94 \times 10^6 \text{ psi}$$
$$G_{12} = 6.10 \times 10^6 \text{ psi}, \quad G_{13} = 3.71 \times 10^6 \text{ psi}$$
$$G_{23} = 6.19 \times 10^6 \text{ psi}, \quad \nu_{12} = 0.44$$

Material II

$$E_1 = 19.2 \times 10^6 \text{ psi}, \quad E_2 = 1.56 \times 10^6 \text{ psi}$$
$$G_{12} = G_{13} = 0.82 \times 10^6 \text{ psi}, \quad G_{23} = 0.523 \times 10^6 \text{ psi}$$
$$\nu_{12} = 0.24.$$

Tables 1–4 contain center deflections $\bar{w}$ while Tables 5–8 contain non-dimensionalized axial stresses $\bar{\sigma}_{11}$ for orthotropic and symmetric cross-ply (0°/90°/0°) plates.

Table 1. Center deflections of orthotropic plates (material I)

			$\bar{w}$					
a/b	h/a	Loading	SS	CC	FF	SC	SF	CF
3	0.2	UN	6.29	3.19	224.4	4.38	50.14	17.55
		TR	9.60	5.29	289.2	6.98	66.30	24.49
		PL	20.95	14.46	371.4	17.09	93.46	40.99
	0.14	UN	14.23	5.89	593.1	8.74	124.72	40.13
		TR	21.15	9.71	761.9	13.71	163.50	55.13
		PL	42.38	25.71	966.8	31.83	222.8	87.19
4	0.2	UN	2.72	1.53	226.3	2.03	34.64	8.07
		TR	4.47	2.68	291.6	3.44	46.13	11.91
		PL	12.38	9.10	374.6	10.53	67.12	23.69
	0.14	UN	5.70	2.66	599.1	3.76	83.60	17.53
		TR	9.14	4.66	769.6	6.32	110.34	25.36
		PL	23.36	15.59	976.6	18.66	154.51	47.13
5	0.2	UN	1.46	0.88	227.1	1.14	25.97	4.29
		TR	2.52	1.59	292.8	2.01	34.76	6.71
		PL	8.39	6.32	376.1	7.27	51.86	15.77
	0.14	UN	2.85	1.49	601.8	2.03	60.68	8.87
		TR	4.84	2.70	773.2	3.56	80.51	13.58
		PL	15.15	10.69	981.3	12.57	115.45	29.86

$\bar{w} = [w(a/2, 0)/q_0]10^6$, $a = 200$ in.

Table 2. Center deflections for cross-ply (0°/90°/0°) laminates (material I)

			$\bar{w}$					
a/b	h/a	Loading	SS	CC	FF	SC	SF	CF
3	0.2	UN	6.85	3.86	215.9	5.10	47.67	18.86
		TR	10.23	6.18	277.7	7.87	62.82	25.97
		PL	20.61	14.92	354.5	17.34	87.27	47.32
	0.14	UN	14.88	6.90	585.5	9.81	121.06	41.87
		TR	21.80	11.08	751.3	15.05	158.35	57.09
		PL	41.18	26.33	949.4	31.99	213.4	87.32
4	0.2	UN	3.12	1.87	217.8	2.43	32.34	9.03
		TR	4.99	3.19	280.2	4.00	42.98	13.06
		PL	12.47	9.48	357.6	10.85	61.72	24.09
	0.14	UN	6.23	3.21	591.4	4.38	80.07	18.88
		TR	9.78	5.47	758.9	7.18	105.47	26.99
		PL	23.01	16.16	959.1	19.00	146.04	47.58
5	0.2	UN	1.73	1.08	218.7	1.38	23.78	4.95
		TR	2.91	1.90	281.3	2.37	31.80	7.53
		PL	8.64	6.65	359.1	7.59	46.99	16.11
	0.14	UN	3.23	1.82	594.2	2.41	57.29	9.84
		TR	5.36	3.21	762.6	4.13	75.91	14.81
		PL	15.19	11.21	963.8	12.97	107.72	30.33

Table 3. Center deflections of orthotropic plates (material II)

			$\bar{w}$					
a/b	h/a	Loading	SS	CC	FF	SC	SF	CF
3	0.2	UN	56.64	33.66	382.8	43.48	216.3	115.82
		TR	81.31	50.85	498.8	63.95	285.8	157.07
		PL	142.14	101.17	662.8	119.01	397.3	236.5
	0.14	UN	120.36	60.18	847.8	82.97	478.6	243.1
		TR	169.24	90.13	1098.3	120.35	626.9	325.6
		PL	282.2	177.64	1435.7	218.3	850.6	475.0
4	0.2	UN	26.89	16.68	383.7	21.33	175.32	66.66
		TR	41.26	26.96	499.8	33.52	232.5	93.14
		PL	86.36	64.65	664.1	74.75	328.6	153.62
	0.14	UN	52.80	28.52	850.1	38.32	386.2	135.05
		TR	79.46	45.89	1101.3	59.63	507.6	185.77
		PL	160.49	110.86	1439.3	131.74	698.4	295.6
5	0.2	UN	15.13	9.62	384.1	12.21	144.3	40.13
		TR	24.51	16.29	500.4	20.18	192.14	58.14
		PL	59.88	45.54	664.8	52.41	275.8	106.45
	0.14	UN	27.84	16.20	851.3	21.24	316.2	78.34
		TR	44.48	27.40	1102.8	34.89	416.9	111.54
		PL	106.94	77.97	1441.3	91.01	581.7	197.38

Table 4. Center deflections for cross-ply (0°/90°/0°) laminates (material II)

a/b	h/a	Loading	$\bar{w}$					
			SS	CC	FF	SC	SF	CF
3	0.2	UN	46.33	26.80	434.7	35.17	236.5	104.16
		TR	68.83	42.18	567.3	53.70	313.1	143.35
		PL	131.74	93.63	757.4	110.40	438.7	225.9
	0.14	UN	96.52	47.57	933.3	66.08	511.4	216.9
		TR	140.12	74.19	1211.0	99.47	671.3	294.0
		PL	255.9	163.82	1591.2	200.1	918.8	446.8
4	0.2	UN	21.61	13.01	435.4	16.90	191.72	55.89
		TR	34.48	21.85	568.1	27.62	254.7	80.03
		PL	80.41	59.17	758.4	69.05	362.4	141.88
	0.14	UN	41.46	22.42	935.1	30.20	414.0	112.47
		TR	64.97	37.48	1213.3	48.90	545.1	158.04
		PL	147.42	102.57	1594.1	121.85	756.1	269.3
5	0.2	UN	12.15	7.36	435.7	9.56	157.83	32.26
		TR	20.39	12.88	568.5	16.35	210.4	48.31
		PL	55.74	40.91	758.9	47.89	303.8	97.17
	0.14	UN	21.93	12.69	936.0	16.72	339.6	62.73
		TR	36.41	22.14	1214.4	28.45	448.6	92.10
		PL	99.02	71.60	1595.6	84.08	630.7	178.91

Table 5. Axial center stresses for orthotropic plates (material I)

a/b	h/a	Loading	$\bar{\sigma}_{11}$					
			SS	CC	FF	SC	SF	CF
3	0.2	UN	1.017	0.425	19.50	0.649	4.141	0.849
		TR	1.975	1.110	26.40	1.448	6.353	2.060
		PL	11.735	10.209	45.12	10.844	18.641	12.997
	0.14	UN	2.058	0.751	39.02	1.187	8.414	1.395
		TR	3.847	1.898	52.43	2.582	12.626	3.445
		PL	18.646	15.184	83.87	16.522	31.92	19.81
4	0.2	UN	0.585	0.253	19.49	0.393	2.472	0.098
		TR	1.204	0.696	26.39	0.913	4.035	0.878
		PL	9.397	8.302	45.17	8.792	14.879	10.494
	0.14	UN	1.132	0.426	39.07	0.679	5.178	0.022
		TR	2.243	1.130	52.52	1.544	8.168	1.291
		PL	14.316	11.945	84.09	12.914	24.93	15.382
5	0.2	UN	0.399	0.184	19.47	0.284	1.543	−0.078
		TR	0.845	0.505	26.37	0.663	2.688	0.466
		PL	7.862	6.957	45.18	7.385	12.354	8.995
	0.14	UN	0.739	0.292	39.06	0.468	3.383	−0.284
		TR	1.504	0.780	52.52	1.071	5.599	0.568
		PL	11.582	9.790	84.17	10.563	20.37	12.880

$\bar{\sigma}_{11} = [\sigma_{11}(a/2, 0, h/2)/q_0]10^6$, $a = 200$ in.

Table 6. Axial center stresses for cross-ply (0°/90°/0°) laminates (material I)

			$\bar{\sigma}_{11}$					
a/b	h/a	Loading	SS	CC	FF	SC	SF	CF
3	0.2	UN	1.082	0.482	19.66	0.729	4.272	1.098
		TR	2.115	1.262	26.51	1.619	6.513	2.402
		PL	11.160	9.792	43.66	10.386	17.708	12.419
	0.14	UN	2.106	0.794	39.54	1.262	8.519	1.622
		TR	3.983	2.077	53.01	2.784	12.766	3.799
		PL	17.928	14.814	82.76	16.049	30.78	19.17
4	0.2	UN	0.620	0.289	19.67	0.439	2.602	0.263
		TR	1.305	0.802	26.52	1.032	4.213	1.138
		PL	9.072	8.082	43.74	8.546	14.181	10.054
	0.14	UN	1.156	0.455	39.62	0.725	5.278	0.185
		TR	2.350	1.260	53.13	1.695	8.319	1.586
		PL	13.949	11.835	83.03	12.735	24.05	14.967
5	0.2	UN	0.427	0.215	19.66	0.321	1.673	0.027
		TR	0.927	0.588	26.51	0.756	2.879	0.662
		PL	7.725	6.890	43.75	7.298	11.859	8.680
	0.14	UN	0.759	0.321	39.63	0.507	3.492	−0.175
		TR	1.593	0.883	53.16	1.190	5.773	0.808
		PL	11.476	9.863	83.14	10.592	19.74	12.645

Table 7. Axial center stresses for orthotropic plates (material II)

			$\bar{\sigma}_{11}$					
a/b	h/a	Loading	SS	CC	FF	SC	SF	CF
3	0.2	UN	2.278	1.242	21.06	1.679	11.352	5.934
		TR	4.879	3.334	29.36	4.002	16.724	9.691
		PL	27.87	25.22	59.66	26.41	43.31	34.21
	0.14	UN	4.700	2.026	40.47	3.020	22.24	10.878
		TR	9.082	5.235	55.33	6.714	31.78	17.068
		PL	43.07	36.90	102.1	39.41	72.08	53.15
4	0.2	UN	0.896	0.510	21.09	0.682	8.854	3.145
		TR	2.647	1.923	29.39	2.256	13.324	5.865
		PL	23.38	21.52	59.71	22.41	38.29	28.41
	0.14	UN	1.759	0.791	40.56	1.170	17.610	5.627
		TR	4.583	2.908	55.45	3.600	25.56	9.957
		PL	34.83	31.03	102.2	32.72	63.17	42.75
5	0.2	UN	0.440	0.262	21.10	0.345	6.821	1.667
		TR	1.699	1.266	29.40	1.471	10.544	3.732
		PL	20.51	18.85	59.73	19.65	34.00	24.64
	0.14	UN	0.813	0.391	40.59	0.570	13.933	2.899
		TR	2.769	1.877	55.50	2.272	20.59	6.111
		PL	29.90	27.00	102.3	28.36	55.78	36.33

Table 8. Axial center stresses for cross-ply (0°/90°/0°) laminates (material II)

a/b	h/a	Loading	$\bar{\sigma}_{11}$ SS	CC	FF	SC	SF	CF
3	0.2	UN	1.541	0.815	22.20	1.118	11.447	4.856
		TR	3.902	2.668	31.11	3.205	16.995	8.389
		PL	28.60	25.93	64.92	27.15	46.21	34.86
	0.14	UN	3.264	1.371	42.32	2.066	22.48	9.106
		TR	7.191	4.197	58.07	5.358	32.28	14.875
		PL	43.54	37.75	109.7	40.19	76.29	53.58
4	0.2	UN	0.580	0.333	22.22	0.441	8.825	2.330
		TR	2.117	1.525	31.13	1.796	13.391	4.821
		PL	24.01	21.88	64.96	22.90	40.62	28.91
	0.14	UN	1.158	0.536	42.39	0.781	17.763	4.300
		TR	3.607	2.327	58.15	2.866	25.88	8.243
		PL	35.44	31.54	109.8	33.32	66.75	43.15
5	0.2	UN	0.291	0.178	22.22	0.230	6.675	1.145
		TR	1.366	0.997	31.14	1.168	10.429	3.001
		PL	20.93	18.84	64.97	19.84	35.81	25.05
	0.14	UN	0.539	0.273	42.41	0.387	13.935	2.063
		TR	2.190	1.496	58.18	1.807	20.69	4.953
		PL	30.30	27.09	109.9	28.61	58.71	36.82

The following notation has been used throughout the tables:

SS—simply supported at $y = -b/2$ and at $y = b/2$;
CC—clamped at $y = -b/2$ and at $y = b/2$;
FF—free at $y = -b/2$ and at $y = b/2$;
SC—simply supported at $y = -b/2$ and clamped at $y = b/2$;
SF—simply supported at $y = -b/2$ and free at $y = b/2$;
CF—clamped at $y = -b/2$ and free at $y = b/2$;
UN—uniformly distributed load;
TR—triangular distributed load;
PL—point load at the center of the plate.

To show the effect of transverse shear strains on the deflections plots of non-dimensionalized center deflection, $\bar{w} = 10^3 w(a/2, 0)h^3E_2/(q_0a^4)$, vs side to thickness ratio of various plates are presented in Figs 3–5. The shear deformation effect is more significant in cross-ply plates than in orthotropic plates. Also, the first-order shear deformation theory (FSDT) over predicts deflections relative to the higher-order theory (HSDT).

Figures 6 and 7 contain plots of the transverse stresses σ_{13} through laminate thickness for various boundary conditions. The stresses were computed using lamina constitutive relations. The transverse shear stresses are constant and parabolic, through thickness of each lamina, respectively, for the first- and higher-order theories. The discontinuity at interface of lamina is due to the mismatch of the material properties. When the stresses (σ_x, σ_y, σ_{xy}) obtained from the constitutive equations are substituted into the equilibrium equations of elasticity and integrated to determine the transverse shear stresses, the resulting functions will be continuous through the thickness.

Plots of the non-dimensionalized center stress, $\bar{\sigma}_{11} = 10^2\sigma_{11}(a/2, 0, h/2)h^2/(q_0a^2)$, vs side to thickness ratio for simply supported and free–free (SSFF) plates are shown in Figs 8 and 9. The shear deformation effect is quite significant for a/h ratios smaller than 10.

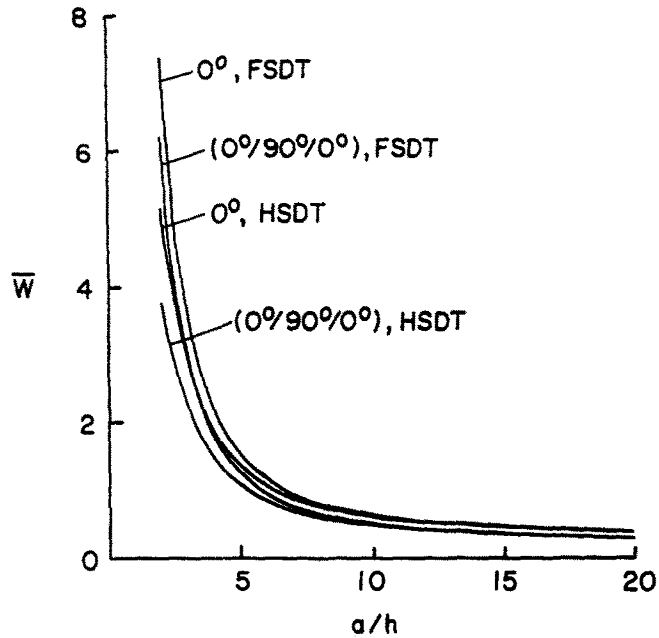

Fig. 3. Non-dimensionalized center deflection vs side to thickness ratio of SSSC plates using the first- and higher-order theories (material II, $a/b = 4$, uniform load).

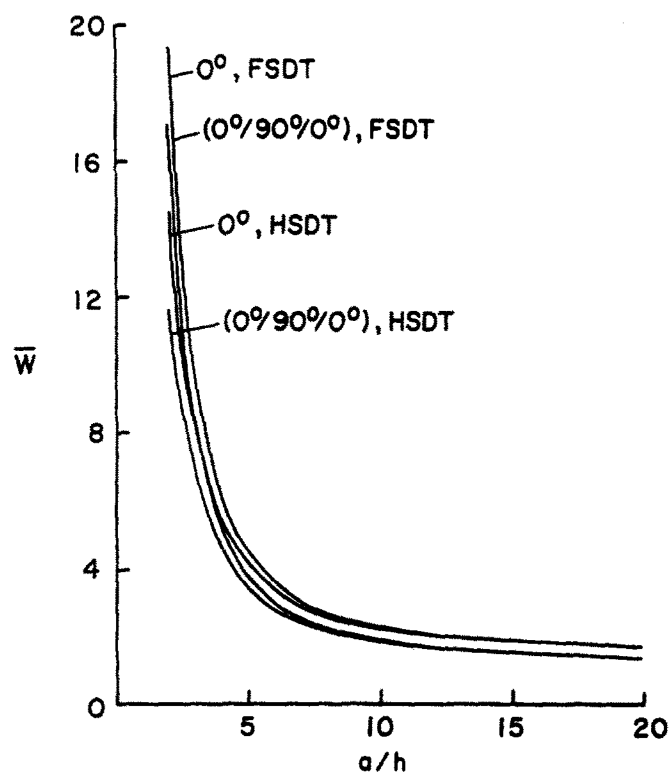

Fig. 4. Non-dimensionalized center deflection vs side to thickness ratio of SSFC laminates using the first- and higher-order theories (material II, $a/b = 4$, uniform load).

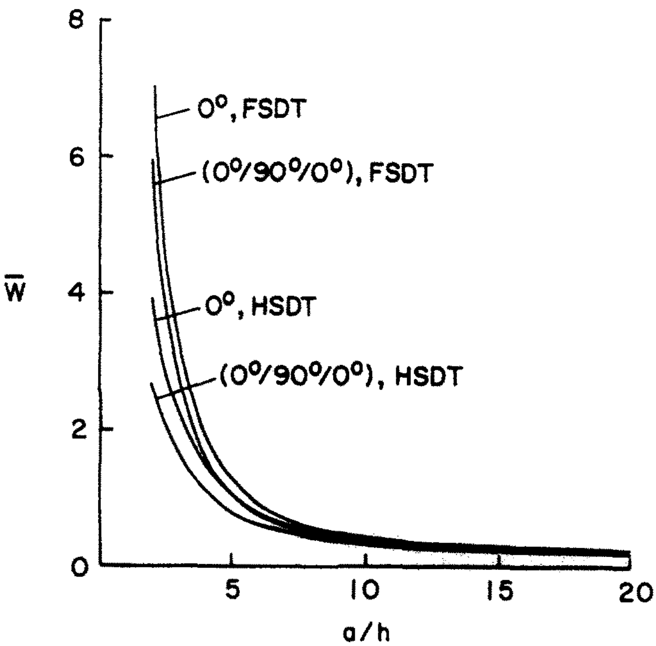

Fig. 5. Non-dimensionalized center deflection vs side to thickness ratio of SSCC laminates using the first- and higher-order theories (material II, $a/b = 4$, uniform load).

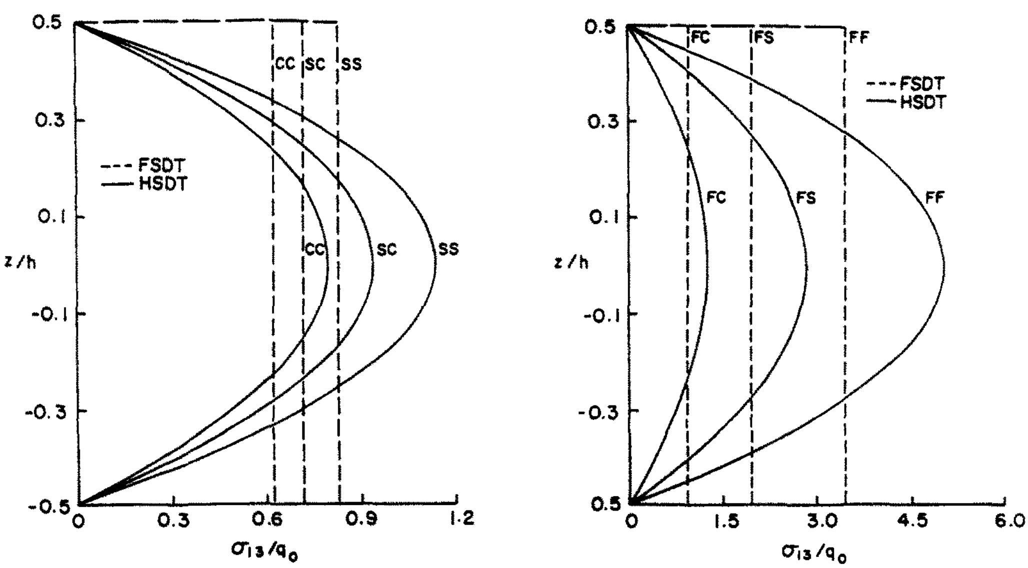

Fig. 6. Variation of the transverse stress through the thickness of orthotropic plates under uniform load and subjected to various boundary conditions (material II, $a/b = 4$, $h/a = 0.14$).

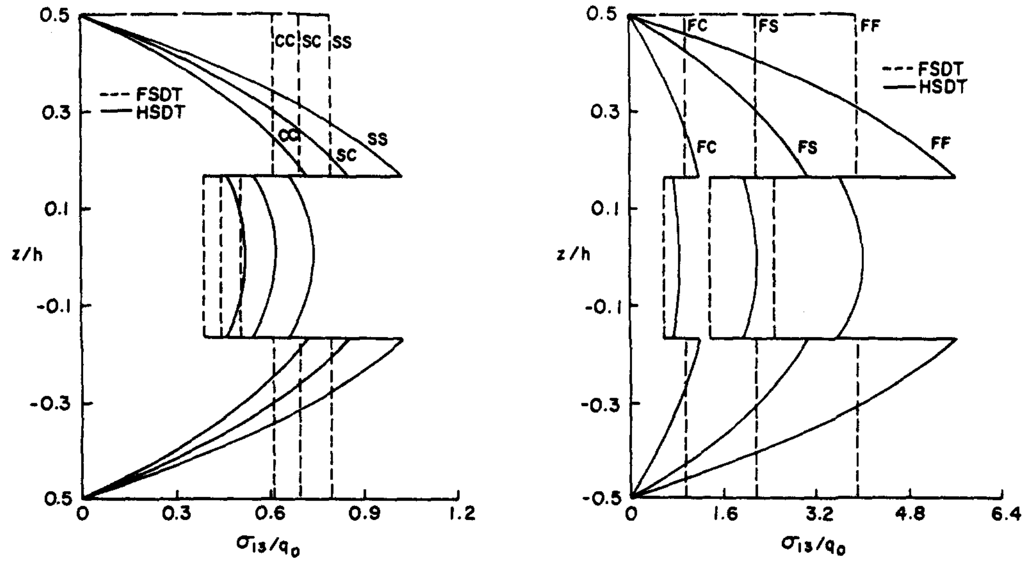

Fig. 7. Variation of the transverse sheer stress through the thickness of cross-ply (0°/90°/0°) laminates under uniform load and subjected to various boundary conditions ($a/b = 4$, material II, $h/a = 0.14$).

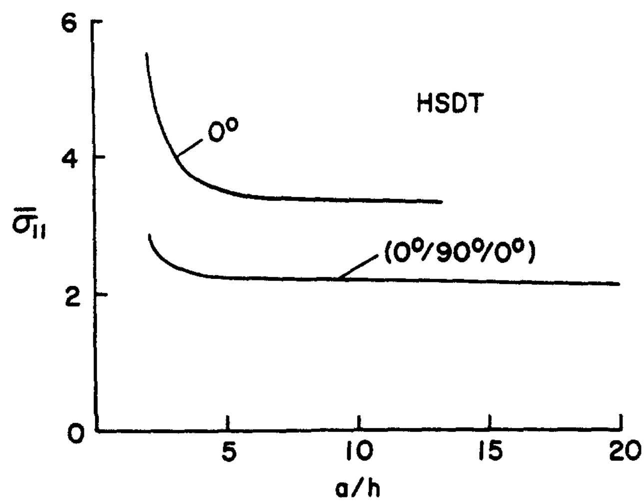

Fig. 8. Non-dimensionalized center stress vs side to thickness ratio for simply supported laminates under uniform load ($a/b = 4$, material II).

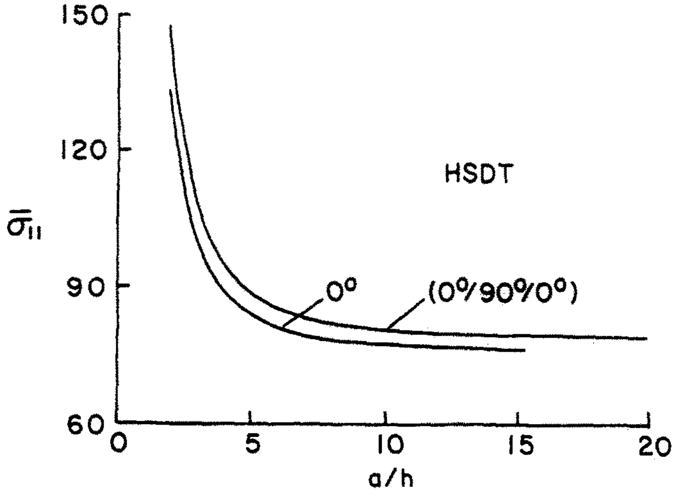

Fig. 9. Non-dimensionalized center stress vs side to thickness ratio for SSFF laminates under uniform load ($a/b = 4$, material II).

CONCLUSIONS

Analytical solutions based on a refined shear deformation plate theory are developed for orthotropic and symmetric cross-ply laminates under various boundary conditions and loads. The Lévy solution method in conjunction with the state-space approach is used to solve the equations. Rectangular plates with simply supported boundary conditions on two parallel edges, while the other two edges are subjected to a combination of free simply supported and clamped boundary conditions are solved. The numerical and graphical results presented should serve as a reference to designers and numerical analysts.

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